Lattice QCD with Applications to B Physics

Lecture 2

Recapitulation

Lattice Field Theory for Scalar Bosons

Lattice Gauge Symmetry

Lattice Gauge Theory

Chiral Symmetry

Lattice Fermions

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Recapitulation of Lecture 1

Flavor physics (and other areas of particle, nuclear, and astro physics) need non-perturbative calculations in QCD.

Need tool(s) that get at the non-perturbative dynamics. Although unitarity, analyticity, symmetry, and renormalization theory are wonderful, they're not enough.

Lattice gauge theory lends a clear definition of the functional integral

$$\langle \bullet \rangle = \frac{1}{Z} \int \mathcal{D}\phi \bullet e^{-S} \quad (\text{with} \quad \langle 1 \rangle \equiv 1)$$

and, thus, provides new tools.

The tool in widest use is a numerical evaluation of the functional integrals, using Monte Carlo methods with importance sampling. The basic idea can be understood from examples in quantum mechanics.

Quantum Field Theory

Today we will go over lattice field theories.

For (both scalar and gauge) bosons the Metropolis method (covered last time) and similar techniques (heat bath, successive overrelaxation) are still used.

Simplicity follows from the locality of interactions in particle physics and the positivity of any unitary action.

Fermions are (numerically) more complicated. The underlying reason for this is the Pauli exclusion principle. The computations are much, much more demanding. As a consequence the necessary algorithms have been the subject of much research and are now too baroque to discuss here.

We shall also see that fermions are theoretically more complicated too.

Lattice Field Theory

(Subset of) $\mathbb{Z}^d$ lattice, spacing a.

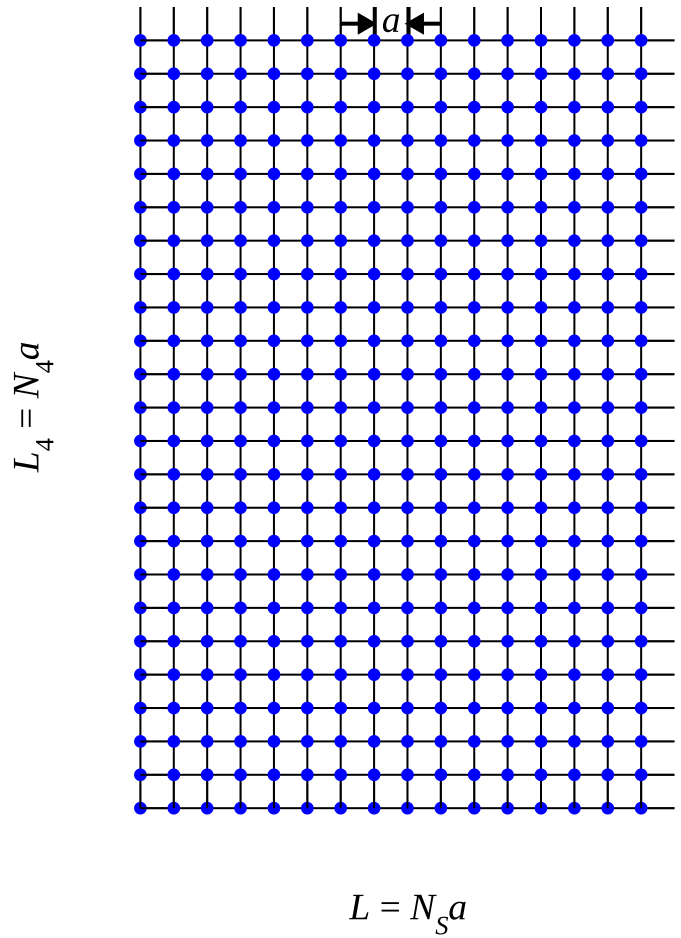
Finite spatial volume, usually with all three sides of (physical length) $L = N_S a$.

Finite time extent, $L_4 = N_4 a$; related to temperature: $k_B T = \hbar c / L_4$.

Boundary conditions usually periodic or anti-periodic, sometimes fixed.

Matter fields on sites.

Gauge fields on links.



Correlation Functions

Recall the large-time behavior of 2-point correlation functions:

$$\int d^3x \, \langle Q_1(\boldsymbol{x}, x_4) Q_2(\mathbf{0}, 0)\rangle_c \stackrel{\text{large } T}{\longrightarrow} \langle 0|\hat{Q}_1(\boldsymbol{p}=\mathbf{0})|1\rangle\langle 1|\hat{Q}_2(\mathbf{0})|0\rangle e^{-m_1 x_4} + \langle 0|\hat{Q}_2(\mathbf{0})|1\rangle\langle 1|\hat{Q}_1(\mathbf{0})|0\rangle e^{-m_1(T-x_4)}.$$

Today we will use this a couple of times to uncover what states the lattice gauge theories contain.

I'll conjure up the momentum space propagator, and Fourier transform from $(\boldsymbol{p}, p_4)$ back to $(\boldsymbol{p}, x_4)$.

The x_4 dependence will allow us to read off the energies; the pattern of matrix elements will also tell us something about the states.

Quantum Mechanics on a Lattice

In quantum mechanics the classcial action (with $\tau = it$)

$$S = \int d\tau \left[\tfrac{1}{2} m \left(\frac{dx}{d\tau} \right)^2 + V(x) \right].$$

We saw last time that the velocity $dx/d\tau$ first appeared in a discrete approximation:

$$S = a \sum_{n=0}^{N-1} \left[\tfrac{1}{2} m \left(\frac{x_{n+1} - x_n}{a} \right)^2 + V(x_n) \right], \quad \mathcal{D}x = \prod_n dx_n.$$

where $n \in \mathbb{Z}$ and dimensionful $\tau = na$.

For field theory we simply repeat this replacement in all d dimensions.

Scalar Field Theory on a Lattice

For a scalar boson field, the action is

$$S = \int d^4x \left[\frac{1}{2} \sum_{\mu=1}^{d} \left(\frac{\partial \phi}{\partial x_\mu} \right)^2 + V(\phi) \right],$$

where $V(\phi) = \frac{1}{2} m_0^2 \phi^2 + \lambda_0 \phi^4/4! + \cdots$.

To obtain the simplest lattice action, replace the derivatives with finite differences in all d directions,

$$S = a^d \sum_n \left[\sum_{\mu=1}^{d} \frac{1}{2} a^{-2} \left(\phi_{n+e^{(\mu)}} - \phi_n \right)^2 + V(\phi_n) \right], \quad \mathcal{D}\phi = \prod_n d\phi_n.$$

where $e^{(\mu)}$ is a unit vector in the μ direction, and $n \in \mathbb{Z}^d$ (or finite subset thereof).

This is a square, cubic, or hypercubic lattice for $d = 2, 3, 4$.
Dimensionful spacetime labels $x_\mu = n_\mu a$.

Free Lattice Field Theory

The kinetic terms may be re-written

$$\sum_n \left(\phi_{n+e^{(\mu)}} - \phi_n\right)^2 = \sum_n 2\phi_n^2 - 2\phi_n\phi_{n+e^{(\mu)}} = \sum_n 2\phi_n^2 - \phi_n\left(\phi_{n+e^{(\mu)}} + \phi_{n-e^{(\mu)}}\right)$$

analogous to integrating by parts $(\partial_\mu\phi)^2 \longrightarrow -\phi\partial_\mu^2\phi$.

So we will take the lattice Lagrangian (or Lagrange density) to be

$$\mathcal{L}_n = a^{-2}\,\frac{1}{2}\sum_\mu \phi_n(t_\mu + t_{-\mu} - 2)\phi_n - V(\phi_n),$$

$$t_{\pm\mu}\phi_n = \phi_{n\pm e^{(\mu)}}, \qquad S = -a^d\sum_n \mathcal{L}_n.$$

For free fields $V(\phi) = \frac{1}{2}m_0^2\phi^2$. In momentum space, the propagator is

$$G(p)^{-1} = \hat{p}^2 + m_0^2,$$

where $\hat{p}^2a^2 = \sum_\mu \frac{1}{2}(e^{ip_\mu a} + e^{-ip_\mu a} - 2) = \sum_\mu[2\sin(\frac{1}{2}p_\mu a)]^2 = \sum_\mu \hat{p}_\mu^2 a^2$.

Fourier transform from p_4 back to x_4:

$$G(\boldsymbol{p},x_4) = \int \frac{dp_4}{2\pi} \frac{e^{ip_4x_4}}{\hat{p}_4^2 + \hat{\boldsymbol{p}}^2 + m_0^2} = \int \frac{dp_4}{2\pi} \frac{a^2 e^{ip_4x_4}}{2 + a^2(\hat{\boldsymbol{p}}^2 + m_0^2) - 2\cos p_4 a}$$

$$= \oint \frac{dz}{2\pi i} \frac{a z^{|x_4|/a}}{2z\cosh Ea - 2(z^2+1)}, \qquad z = e^{\mathrm{sign}(x_4) i p_4 a}$$

$$= \frac{a e^{-E|x_4|}}{2\sinh Ea} \qquad \text{expected} \quad \frac{e^{-(\boldsymbol{p}^2+m^2)^{1/2}|x_4|}}{2(\boldsymbol{p}^2+m^2)^{1/2}}$$

where $\cosh Ea = 1 + \frac{1}{2}a^2(\hat{\boldsymbol{p}}^2 + m_0^2)$.

Here we have defined the energy through the fall-off of the correlation function:

$$\langle \phi(\boldsymbol{p},x_4)\phi(\boldsymbol{q},x_4)\rangle_c = (2\pi)^{d-1}\delta(\boldsymbol{p}-\boldsymbol{q})G(\boldsymbol{p},x_4) \quad \text{or} \quad L^{1-d}\delta_{\boldsymbol{p},\boldsymbol{q}}G(\boldsymbol{p},x_4)$$

We see that discretization effects distort the energy.
They also change the normalization so it is no longer canonical.

Lattice Gauge Symmetry

Now suppose we have a complex (*i.e.*, charged) scalar field.

Suppose it transforms under some gauge group as

$$\phi(y) \mapsto g(y)\phi(y), \qquad \phi^\dagger(x) \mapsto \phi^\dagger(x)g^{-1}(x), \qquad g^{-1} = g^\dagger.$$

In the Lagrangian for scalar fields, the local terms are automatically gauge invariant if the potential is a function $\phi^\dagger\phi$ (as it would be):

$$\phi^\dagger(x)\phi(x) \mapsto \phi^\dagger(x)g^{-1}(x)g(x)\phi(x) = \phi^\dagger(x)\phi(x).$$

Not so for the kinetic terms, which involve fields on different sites:

$$\phi^\dagger(x)\phi(y) \mapsto \phi^\dagger(x)g^{-1}(x)g(y)\phi(y),$$

which is not (yet) gauge invariant.

Suppose we had an object that transforms as

$$U(x,y) \mapsto g(x)U(x,y)g^{-1}(y),$$

then $\phi^\dagger(x)U(x,y)\phi(y)$ is gauge-invariant.

From continuum gauge-field theory, we have a suitable object

$$U(x,y) = \mathrm{P}\exp\left(\int_y^x dz \cdot A\right),$$

ordered along some path from y to x. Verify $U(x,y) \mapsto g(x)U(x,y)g^{-1}(y)$.

The $U(x,y)$ are often called parallel transporters (by mathematicians) or Wilson lines (by physicists).

We can make $\phi^\dagger(x)t_{\pm\mu}\phi(x)$ gauge invariant simply by inserting the parallel transporter along the link: $t_{\pm\mu}\phi(x) \to T_{\pm\mu}\phi(x) = U(x, x \pm ae^{(\mu)})\phi(x \pm ae^{(\mu)})$.

Lattice Gauge Fields

This means that the basic variables for lattice gauge fields are

$$U_\mu(x) = U(x, x + ae^{(\mu)}), \qquad U(x, x - ae^{(\mu)}) = U^\dagger(x - ae^{(\mu)}, x) = U_\mu^\dagger(x - ae^{(\mu)})$$

They take values in a Lie group. A_μ takes values in the Lie algebra.

So integrating over all $U_\mu(x)$ sums over all possible lattice gauge fields.

What is the measure? We want

$$\int dU\, f(U) = \int dU\, f(gU) = \int dU\, f(Ug^{-1}) \Rightarrow \int dU\, U = 0$$

$$\int dU\, 1 = 1 \Rightarrow \int dU\, U_{ij} U_{lk}^* = \frac{1}{N} \delta_{il} \delta_{kj} \quad \text{for SU}(N)$$

Mathematicians call this Haar measure.

The functional integral is then $\mathcal{D}U = \prod_{n,\mu} dU_\mu(n)$.

Lattice Gauge Action

Now we need an action for lattice gauge fields.

The simplest one is obtained by analogy with the simplest action for scalar fields. Now the translations are

$$T_\mu U_\nu(x) = U_\mu(x) U_\nu(x + ae^{(\mu)}) U_\mu^\dagger(x + ae^{(\nu)})$$
$$T_{-\mu} U_\nu(x) = U_\mu^\dagger(x - ae^{(\mu)}) U_\nu(x - ae^{(\mu)}) U_\mu(x - ae^{(\mu)} + ae^{(\nu)})$$

So,

$$-\sum_{x,\mu} \mathrm{tr}[U_\nu^\dagger(x)(T_\mu + T_{-\mu} - 2)U_\nu(x)] = 2\sum_{x,\mu} P_{\mu\nu}$$

$$P_{\mu\nu} = \mathrm{Re}\,\mathrm{tr}[1 - U_\mu(x) U_\nu(x + ae^{(\mu)}) U_\mu^\dagger(x + ae^{(\nu)}) U_\nu^\dagger(x)].$$

Wilson proposed the action

$$S = \frac{\beta}{2N} \sum_{x,\mu,\nu} P_{\mu\nu}(x) \qquad \text{the Wilson (plaquette) action}$$

Lattice Gauge Actions

Remarkably, the plaquette action reduces to the Yang-Mills action

$$S = -\frac{1}{2g_0^2} \int d^4x \, \mathrm{tr}[F_{\mu\nu} F^{\mu\nu}]$$

when $a \to 0$, identifying $\beta = 2N/g_0^2$.

The lattice breaks space-time Lorentz (actually Euclidean) invariance, so there are discretization effects of the form $a^2 \bar{\phi} \partial_\mu^4 \phi$. Being suppressed by a^2, it's tolerable.

Maintaining exact gauge invariance forbids interactions like $m_g^2 A_\mu^a A_\mu^a$ for gluons.

The lattice actions presented so far are simple, but not the only choice. (Think of numerical methods for PDEs.) To maintain gauge invariance, the Lagrangian of lattice gauge theory must be built out of local combinations

$$\mathrm{tr}[U_{\mathcal{C}}(z,z)], \qquad \phi^\dagger(x) U_{\mathcal{P}}(x,y) \phi(y), \qquad \bar{\psi}(x) U_{\mathcal{P}}(x,y) \psi(y),$$

such that the continuum Lagrangian emerges when $a \to 0$.

Chiral Symmetry

The Lagrangian for continuum fermions is

$$\mathcal{L}(x) = -\bar{\psi}(x)\,(\not{D} + m_0)\,\psi(x), \qquad \not{D} = D^\mu \gamma_\mu, \qquad D^\mu = \frac{\partial}{\partial x_\mu} + A^\mu.$$

The field ψ annihilates quarks and creates anti-quarks. It has four components, two each for (spin $\frac{1}{2}$) quarks and anti-quarks.

The matrices γ_μ satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ (in Euclidean metric) and encode the spinor representation of the Lorentz (actually, Euclidean) group.

The Lagrangian is invariant under phase rotations (the group U(1))

$$\psi \mapsto e^{i\theta}\psi, \qquad \bar{\psi} \mapsto \bar{\psi}e^{-i\theta},$$

which corresponds to baryon number. (Also invariant under SU(3) color, but that's not where we're going.)

If there are n_f flavors, then the flavor symmetry is $\mathrm{SU}(n_f) \times \mathrm{U}(1)$.

In studying $\mathcal{L}(x) = -\bar\psi(x)\,(\not{D} + m_0)\,\psi(x)$, there is another "Dirac" matrix of interest $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, which turns out to satisfy $\{\gamma_5, \gamma_\mu\} = 0$, $\gamma_5^\dagger = \gamma_5$, $\gamma_5^2 = 1$.

If the mass vanishes, there is another ("axial") symmetry

$$\psi \mapsto e^{+i\alpha\gamma_5}\psi, \qquad \bar\psi \mapsto \bar\psi e^{+i\alpha\gamma_5}.$$

With n_f flavors the flavor symmetry becomes $\mathrm{SU}_V(n_f) \times \mathrm{SU}_A(n_f) \times \mathrm{U}_V(1)$.

It is sometimes convenient (though not too important here) to introduce

$$\psi_L = \tfrac{1}{2}(1-\gamma_5)\psi, \quad \psi_R = \tfrac{1}{2}(1+\gamma_5)\psi, \quad \bar\psi_L = \bar\psi\tfrac{1}{2}(1+\gamma_5), \quad \bar\psi_R = \bar\psi\tfrac{1}{2}(1-\gamma_5)\psi.$$

Then

$$\mathcal{L}(x) = -\bar\psi_L(x)\not{D}\psi_L(x) - \bar\psi_R(x)\not{D}\psi_R(x) - m_0\bar\psi_L(x)\psi_R(x) - m_0\bar\psi_R(x)\psi_L(x)$$

and the $(m_0 = 0)$ symmetry can be recast as $\mathrm{SU}_L(n_f) \times \mathrm{SU}_R(n_f) \times \mathrm{U}_V(1)$.

Chiral symmetry.

Chiral symmetry plays an important role in QCD.

For massless quarks the vacuum spontaneously breaks

$$\mathrm{SU}_L(n_f) \times \mathrm{SU}_R(n_f) \to \mathrm{SU}_V(n_f)$$

and the up, down, and strange quarks have masses small enough that this spontaneious breaking is evident in the spectrum:

$$m_\pi^2 \ll m_\rho^2, \quad m_K^2 = 0.3 m_{K^*}^2.$$

An effective field theory for the pion cloud: chiral perturbation theory (χPT).

You might wonder why there is no axial $\mathrm{U}_A(1)$, in analogy with $\mathrm{U}_V(1)$. A quantum effect called the “anomaly” breaks it explicitly.

The anomaly is observed experimentally in $\pi_0 \to \gamma\gamma$.

Lattice Fermions

We would like to find a discretization of the Lagrangian

$$\mathcal{L}(x) = -\bar{\psi}(x)\,(\not{D} + m_0)\,\psi.$$

$\not{D}$ is anti-Hermitian, $\not{D}^\dagger = -\not{D}$.

We've learned how to put in the gauge fields, so let's focus on the free case.

The simple choices $\partial_\mu \to t_\mu - 1$ or $1 - t_{-\mu}$ are not anti-Hermitian, so the particle and anti-particle parts of ψ would propagate differently.

The simplest anti-Hermitian choice is $(t_\mu - t_{-\mu})/2a$, yielding the naive action

$$\mathcal{L}_{\mathrm{NF}} = -\bar{\psi}(x)\left[\frac{1}{2a}\sum_\mu \gamma_\mu(t_\mu - t_{-\mu}) + m_0\right]\psi(x)$$

We will now try to see what this is, by looking at the propagator.

Naïve Propagator

The propagator that one gets from this Lagrangian is ($x_4 = n_4 a > 0$)

$$G(\boldsymbol{p}, x_4) = \int \frac{dp_4}{2\pi} e^{ip_4 x_4} \frac{a}{i\sum_\mu \gamma_\mu \sin(p_\mu a) + m_0 a}$$

$$= \frac{1}{\sinh(2Ea)} \left[e^{-Ex_4} \left(\gamma_4 \sinh(Ea) - i\sum_i \gamma_i \sin(p_i a) + m_0 a \right) + (-1)^{n_4} e^{-Ex_4} \left(-\gamma_4 \sinh(Ea) + i\sum_i \gamma_i \sin(p_i a) + m_0 a \right) \right]$$

where $\sinh^2(Ea) = \sum_i \sin^2(p_i a) + (m_0 a)^2$.

The first term is desirable; the second term has a peculiar oscillation $(-1)^{n_4}$.

Moreover, there are low-lying states for $p_i a \sim 0, (\pi, 0, 0), (\pi, \pi, 0), (\pi, \pi, \pi) \cdots$.
$2 \times 8 = 16$ species in all. The Fermion Doubling Problem.

In perturbation theory, 16 species arise from the 16 regions where $\sin(p_\mu a) \sim a$.

Vacuum polarization induces the coupling to run:

$$a\frac{dg_0^2}{da} = -2\frac{g_0^4}{16\pi^2}\left(\frac{11N}{3} - \frac{2n_f}{3}\right) + O(g_0^6)$$

With naïve lattice fermions one finds this usual form of the result, but with $n_f = 16n_\psi$.

Anomaly in flavor-singlet axial current $A_\mu = \frac{1}{2}\bar{\psi}(T_\mu + T_{-\mu})\gamma_\mu\gamma_5\psi$ from gauging

$$\psi \mapsto e^{i\alpha\gamma_5}\psi, \qquad \bar{\psi} \mapsto \bar{\psi}e^{i\alpha\gamma_5}.$$

Normally $\partial \cdot A = 2m\bar{\psi}\gamma_5\psi + \mathcal{A}$, where $\mathcal{A}$ is the axial anomaly. With naïve lattice fermions

$$\mathcal{A}_{\mathrm{NF}} = (1 - 4 + 6 - 4 + 1)\mathcal{A} = 0$$

Recall, the decay $\pi^0 \to \gamma\gamma$ should proceed through it, so that is lost with naïve fermions.

Why does this happen? On lattice, symmetries are either exact or explicitly broken.

The Fermion Doubling Problem

The spectrum, vacuum polarization, and axial anomaly of naïve fermions are the first sign that fermions do not like the lattice.

The Nielsen-Ninomiya Theorem says that there is no *ultra*-local fermion action with the full chiral symmetry, no additional states, and a real, positive transfer matrix.

For a long time "ultra-local" was phrased "local". Ultra-local means that interactions coupling fields vanish if the fields are farther apart than some fixed distance, of order a few lattice spacings.

"Local" means that they merely fall off exponentially.

There are now formulations of lattice fermions with undoubled spectra and full chiral symmetry. Not ultra-local so no transfer matrix.

See P. Hasenfratz, `hep-lat/0406033`, for a succinct history and review.

Wilson Fermions

To cope with the doubled spectrum, Wilson reasoned as follows.

The particle states should use projection matrices so that some components move forward in time, and others backward.

$$\partial_4\psi(x) \to \left(\frac{1-\gamma_4}{2}\frac{t_4-1}{a} + \frac{1+\gamma_4}{2}\frac{1-t_{-4}}{a}\right)\psi(x)$$

$$\bar\psi(x)\gamma_4\partial_4\psi(x) \to \bar\psi(x)\left(\frac{\gamma_4-1}{2}\frac{t_4-1}{a} + \frac{1+\gamma_4}{2}\frac{1-t_{-4}}{a}\right)\psi(x)$$

$$= \bar\psi(x)\left(\gamma_4\frac{t_4-t_{-4}}{2a} - \tfrac{1}{2}a\frac{t_4+t_{-4}-2}{a^2}\right)\psi(x)$$

Repeat in all directions, leading to

$$\mathcal{L}_{\mathrm{WF}} = \mathcal{L}_{\mathrm{NF}} + \tfrac{1}{2}a\sum_\mu \bar\psi(x)\left(\frac{t_\mu+t_{-\mu}-2}{a^2}\right)\psi(x)$$

Propagator for free Wilson fermions (for $x_4 > 0$)

$$G(\boldsymbol{p}, x_4) = \frac{ae^{-Ex_4}}{2\sinh(Ea)} \frac{\gamma_4 \sinh(Ea) - i\sum_i \gamma_i \sin(p_i a) + m_0 a + \frac{1}{2}a^2\hat{\boldsymbol{p}}^2 + 1 - \cosh(Ea)}{1 + m_0 a + \frac{1}{2}a^2\hat{\boldsymbol{p}}^2}$$

with

$$\cosh(Ea) = 1 + \frac{1}{2}\frac{\sum_i \sin^2(p_i a) + (m_0 a + \frac{1}{2}a^2\hat{\boldsymbol{p}}^2)^2}{1 + m_0 a + \frac{1}{2}a^2\hat{\boldsymbol{p}}^2}$$

Now no oscillating state arises, and the energy is low only when $\boldsymbol{p}$ is small.

The price paid is sacrificing chiral symmetry $\psi \mapsto e^{i\alpha\gamma_5}\psi$, $\bar{\psi} \mapsto \bar{\psi}e^{i\alpha\gamma_5}$. Both the mass term and the new Wilson term break chiral symmetry explicitly.

$$m_R = Z_m(g_0^2)\left[m_0 + a^{-1}g_0^2 C(g_0^2)\right].$$

On the other hand, the axial anomaly does come out correctly: the Wilson term washes out the additional poles that generated extra anomalies (to cancel the total).

Staggered Fermions

In the early days, Susskind studied a Hamiltonian lattice gauge theory (discrete space, continuous time, canonical Hamiltonian with conjugate momenta). He found a way to formulate the fermions with less doubling.

On a spacetime lattice start with the naïve fermion Lagrangian, re-written here

$$\mathcal{L}_{\mathrm{NF}} = -\bar{\psi}(n)\left[\frac{1}{2a}\sum_{\mu}\gamma_{\mu}(T_{\mu} - T_{-\mu}) + m_0\right]\psi(n)$$

with $n \in \mathbb{Z}^4$ dimensionless site labels.

Introduce a unitary similarity transformation

$$\psi(n) = \Omega(n)\Psi(n), \qquad \bar{\psi}(n) = \bar{\Psi}(n)\Omega^{\dagger}(n), \quad \Omega(n) = \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}$$

$$\Omega^{\dagger}(n)\gamma_{\mu}T_{\pm\mu}\Omega(n) = \eta_{\mu}(n)T_{\pm\mu}, \qquad \eta_{\mu}(n) = (-1)^{n_1+\cdots n_{\mu-1}}$$

After the transformation, the Lagrangian is

$$\mathcal{L}_{\text{NF}} = -\sum_{\alpha=1}^{4} \bar{\Psi}_\alpha(n) \left[\frac{1}{2a} \sum_\mu \eta_\mu(n)(T_\mu - T_{-\mu}) + m_0 \right] \Psi_\alpha(n)$$

Chiral symmetry remains intact. After the similarity transformation

$$\Omega^\dagger(n)\gamma_5\Omega(n) = \gamma_5\eta_5(n), \qquad \eta_5(n) = (-1)^{n_1+n_2+n_3+n_4} =: \varepsilon(n)$$

$$\Psi(n) \mapsto e^{i\alpha\gamma_5\eta_5(n)}\Psi(n), \qquad \bar{\Psi}(n) \mapsto \bar{\Psi}(n)e^{i\alpha\gamma_5\eta_5(n)}.$$

The chiral transformation rotates even sites ($n_1+n_2+n_3+n_4 \bmod 2 = 0$) one way and odd sites ($n_1+n_2+n_3+n_4 \bmod 2 = 1$) the other, so it is still global.

Written this way, the lattice fermion field has 4 pieces, each with the same Lagrangian; two each with $\gamma_5\Psi = \pm\Psi$. Truncate $\sum_{\alpha=1}^{4}$ to one component.

The Lagrangian for the one component field χ is

$$\mathcal{L}_{\text{stag}} = -\bar{\chi}(n)\left[\frac{1}{2a}\sum_{\mu}\eta_{\mu}(n)(T_{\mu}-T_{-\mu})+m_0\right]\chi(n)$$

with U(1)$\times$U(1) chiral symmetry

$$\chi(n)\mapsto e^{i\theta+i\alpha\eta_5(n)}\chi(n), \qquad \bar{\chi}(n)\mapsto\bar{\chi}(n)e^{-i\theta+i\alpha\eta_5(n)}.$$

This symmetry is enough to forbid an additive counter-term to the bare mass.

Vacuum polarization now behaves as if there are four species. Originally, these were called flavors, in the hope that the four species could be given different masses and correspond to u, d, s, c. Now they are looked at as unphysical and called “tastes”.

The Noether currents corresponding to the U(1)$\times$U(1) chiral symmetry are conserved; the symmetry is exact. Another (non-Noether) current yields the correct anomaly.

Staggered fermions still have the oscillating states.

Ginsparg-Wilson Relation

In continuum gauge theories, chiral symmetry (for physical amplitudes) follows essentially from $\{\not{D}, \gamma_5\} = 0$.

It is worth asking whether this condition is (on a lattice) necessary, or merely sufficient.

It is only sufficient; Ginsparg and Wilson derived the necessary condition:

$$D_{\mathrm{lat}}^{-1}\gamma_5 + \gamma_5 D_{\mathrm{lat}}^{-1} = a\gamma_5 \qquad \Rightarrow \qquad \gamma_5 D_{\mathrm{lat}} + D_{\mathrm{lat}}\gamma_5 = aD_{\mathrm{lat}}\gamma_5 D_{\mathrm{lat}}$$

In the form on the left, we see that in correlation functions the violation of chiral symmetry is a local "contact" term, which drops out of the long-distance physics.

Until a few years ago, no solutions (except in free field theory) were known. Now some local, but not ultra-local, solutions have been found.

Numerical Methods for Fermions

The incorporation of fermions into numerical simulations is the most daunting computational problem in lattice QCD.

The Pauli exclusion principle states that two fermions cannot be in the same state.

Therefore, the integration variables in the functional integral are Grassman numbers:

$$\{\psi_a, \psi_b\} = \{\bar\psi_a, \psi_b\} = \{\bar\psi_a, \bar\psi_b\} = 0$$

The integration rule is (α complex, ε & $\bar\varepsilon$ Grassman) Berezin

$$\int d\psi(\alpha + \bar\varepsilon\psi) = -\bar\varepsilon, \qquad \int d\bar\psi(\alpha + \bar\psi\varepsilon) = \varepsilon$$

Invariance under multiplication says

$$\psi = \xi\psi' \qquad \Rightarrow \qquad d\psi = d\psi'/\xi$$

Every action that we introduced takes the form

$$S = \sum_{a,b} \bar\psi_a M_{ab} \psi_a, \qquad M = M(U).$$

Using the rules of Grassman integration

$$\int \mathcal{D}\psi\mathcal{D}\bar\psi e^{-\bar\psi M\psi} = \int \mathcal{D}\psi'\mathcal{D}\bar\psi' e^{-\bar\psi' V^{-1}MV\psi'} = \prod_a [V^{-1}MV]_{aa} = \det M$$

The physical interpretation of $\det M(U)$ is all possible fermions loops in the background of gauge field U.

A numerical simulation the generates gauge fields with weight $\det M\, e^{-S_{\rm gauge}}$.

This is normal arithmetic, but the computational problem is huge.

M is a $(3\cdot 4\cdot N_S^3\cdot N_4)\times(3\cdot 4\cdot N_S^3\cdot N_4)$ matrix.
(Sparse for naïve, staggered, and Wilson, but not GW; omit 4 for staggered.)

Summary of Fermion Methods

Pattern of chiral symmetry breaking for various formulations of lattice fermions.

formulation	$G \to H$	CPU
continuum QCD	$\mathrm{SU}(n_f) \times \mathrm{SU}(n_f) \to \mathrm{SU}(n_f)$	
staggered[§]	$\Gamma_4 \times \mathrm{U}(1) \to \Gamma_4$	fast: $m_q > 0.1 m_s$, but $n_f = 4$
Wilson	$\mathrm{SU}(n_f) \to \mathrm{SU}(n_f)$	slower: $m_q > 0.5 m_s$
G-W	$\mathrm{SU}(n_f) \times \mathrm{SU}(n_f) \to \mathrm{SU}(n_f)$	slower still: M not sparse

[§] The vector part Γ_4 is a finite (Clifford) group.

For all methods, the computation of $\det M$ (or changes in $\det M$) gets slower and slower as the quark mass decreases (ratio of eigenvalues).

Effective field theories can be used to show that the breaking of $\mathrm{SU}(n_f) \times \mathrm{SU}(n_f)$ is $O(a^2)$ in staggered, and $O(a)$ in (unimproved) Wilson.

Old Pessimism $\longrightarrow$ New Optimism!

It is a good time to take up the study of lattice QCD.

To reduce the computational burden, until now almost all calculations of physically interesting masses or hadronic matrix elements have been done in the so-called "quenched approximation."

This corresponds to omitting all vacuum loops, and compensating the omission with *ad hoc* shifts in the bare gauge coupling and masses.

It's a bit like a dielectric approximation, $e^2 \rightarrow e^2/\varepsilon$. One can only hope that it works when focusing on a narrow range of energies [$\varepsilon(\omega) = \text{constant}$].

For example, many (of my own) papers include statements like "the nth error bar comes from the discrepancy in determining m_b from Υ spectrum instead of the B system."

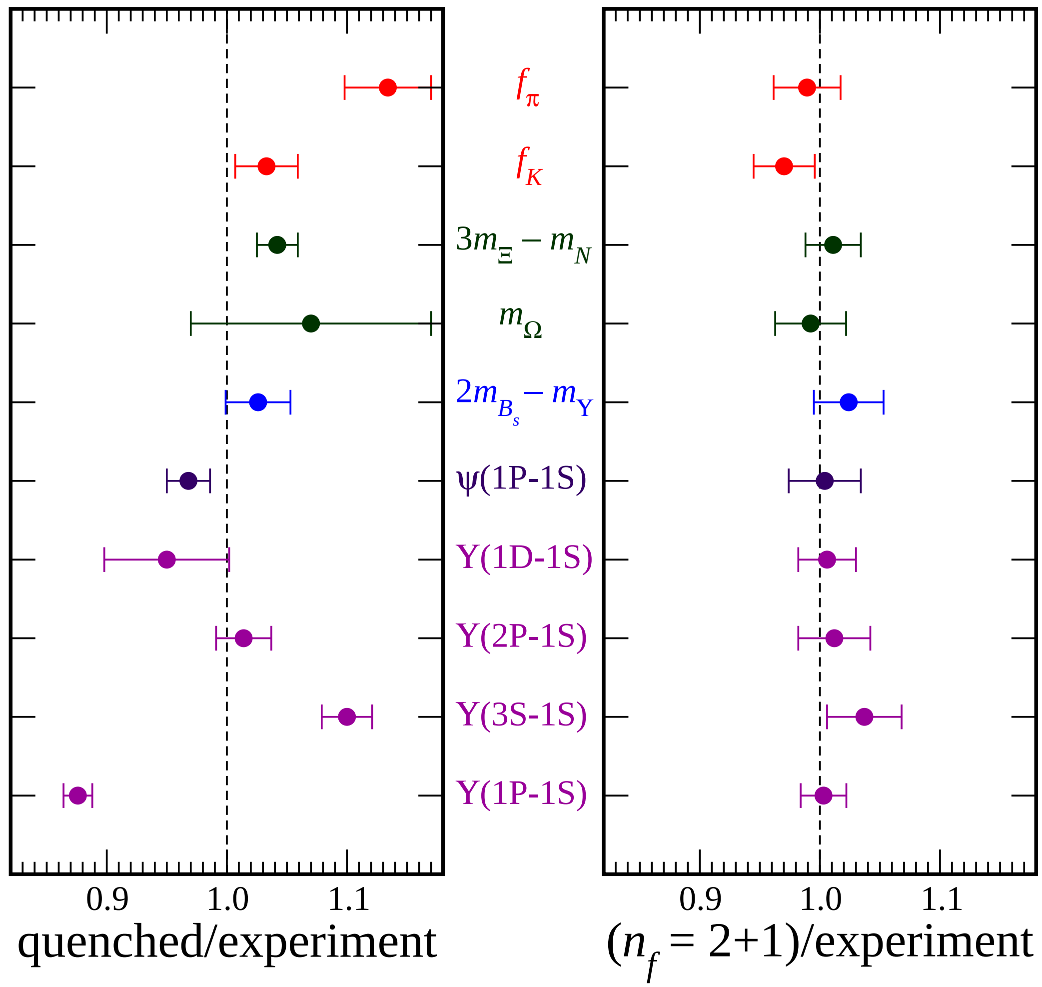
'Twas very unsatisfactory.

In 2003, however, 26 authors produced this plot: $\Longrightarrow$

Set $1+4$ free parameters with $1+4$ meson masses.

Quenched (on left) shows discrepancies as much as 10–15%.

Unquenched QCD (on right) shows discrepancies of a few %—within the error bars.



Davies *et al.*, hep-lat/0304004

The five fiducial quantities ($m_{\Upsilon(2S)} - m_{\Upsilon(1S)}$, m_π^2, m_K^2, m_{D_s}, and $m_{\Upsilon(1S)}$) and the nine shown are all, in a certain sense, "gold-plated."

The gold-plated class includes stable-particle masses and hadronic matrix elements with at most one hadron in the initial or final states

Unstable particles and non-leptonic decays inevitably entail multi-particle states—much more difficult (to be explained later).

This may seem like a disappointing restriction.

There are, however, gold-plated matrix elements for extracting *all* CKM elements $|V_{qq'}|$, except $|V_{tb}|$. (Top quark decays before hadronizing.)

It's not unrealistic to expect the theoretical uncertainty in the CKM matrix to be reduced to a few percent in the next 2–3 years.

Disclaimer

These results obtained with improved staggered quarks in the sea: 2+1 flavors.

Recall that staggered fermions come in four "tastes." The extra degrees of freedom are removed by using $[\det_4(\not{D}_{\text{stag}}+m)]^{1/4}$ instead of $\det_1(\not{D}+m)$.

At non-zero lattice spacing, this prescription leads to violations of unitarity, observed in numerical data for the a_0 propagator.

Conjectured (based on plausibility arguments) to be manageable using "rooted staggered chiral perturbation theory."

No proof, however, and therefore remains controversial. A recent review of these issues deemed rooted staggered quarks to be "ugly" but likely viable in the continuum limit [S.R. Sharpe, hep-lat/0610094].

Other methods of treating the quark sea are 6–12 years behind.